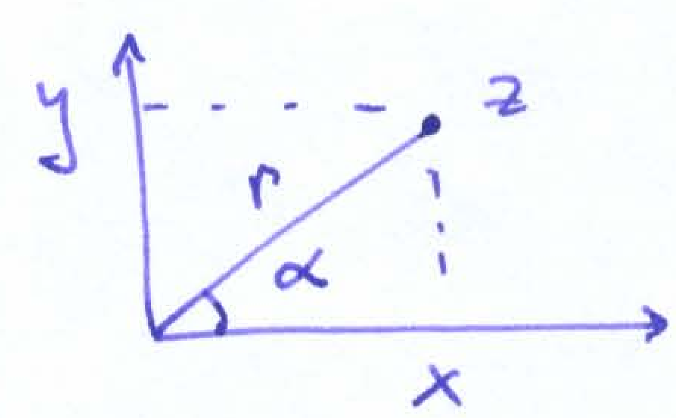
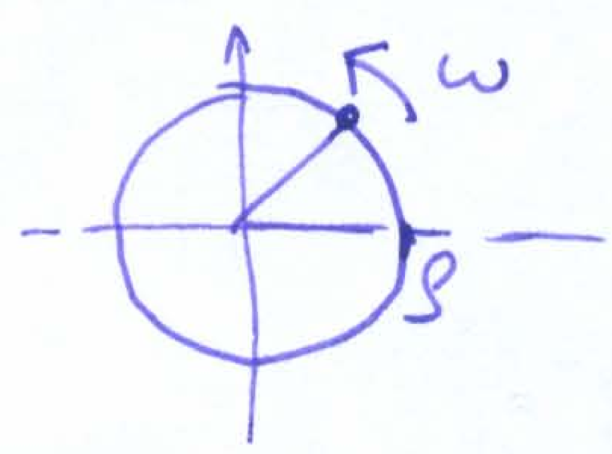
Pedestrian Quantum Mechanics

References : Dirac – Quantum Mechanics (Ch. 1).
Feynman – Lectures on Physics (Vol. III, chs. 7–11)
Baym – Quantum Mechanics (chs. 1–2)
Lipkin – Selected Topics in Quantum Mechanics (Ch. 1)

Reminder :

Complex numbers :





$$z = x + iy = r(\cos\alpha + i \sin\alpha)$$
$$\equiv r \cdot e^{i\alpha}$$

$\rho\, e^{i\omega t}$: rotates around origin, at radius ρ, with angular velocity ω.

Differential equations : (simplest)

$$\frac{d}{dt} x(t) = A\, x(t) \quad \Rightarrow \quad x(t) = (\text{const.}) \times e^{At}$$

$$\frac{d}{dt} z(t) = -\frac{i}{\hbar} E\, z(t) \quad \Rightarrow \quad z(t) = \rho\, e^{-\frac{i}{\hbar} E t}$$
$$|z(t)| = \rho$$

Classical versus Quantum Dynamics

Classical :
- $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \vec{F}(\vec{r}, \vec{v}, t)$

  Given $\vec{r}(t_0)$, $\vec{v}(t_0)$, find $\vec{r}(t)$, $\forall t$.

- In principle, can measure anything
  with arbitrary accuracy,
  with arbitrary small disturbance of the system.
  (exp: position of a billiard ball)

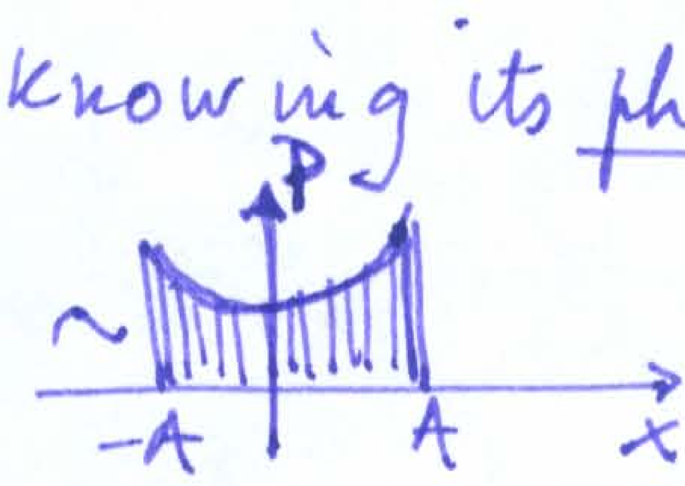
whereas

Quantum :
- Dirac's notion of "small" system :
  disturbed uncontrolably by a measurement
  (exp: position of an electron : photon perturbs it!)

- HOW TO DESCRIBE DYNAMICS THEN ?

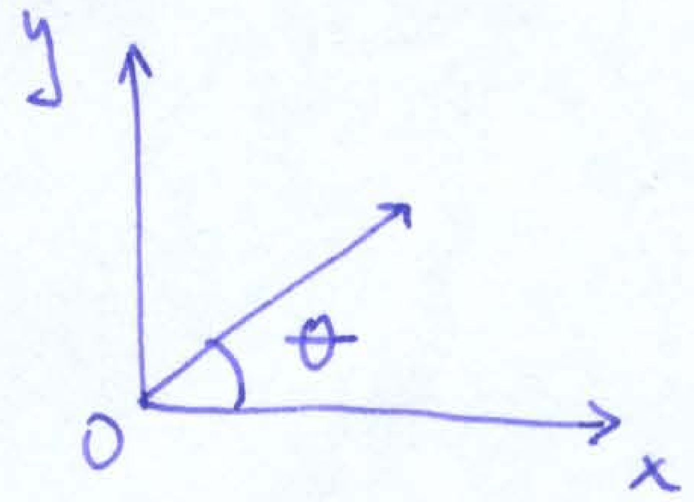
  All one can hope is a statistical description, like giving

  The probability for something to happen

  Imagine an oscillator, without knowing its phase
  Then probability to find particle is ~ 

Polarisation states of a photon

Classical: • Beam of light linearly polarised at angle θ.



• Polariser with axis parallel to Ox.

$$\Downarrow$$

$$I_{out} = I_{in} \cdot \cos^2\theta$$

Quantum: How to interpret that, in terms of INDIVISIBLE PHOTONS ?

(remember photoelectric effect)

0. A photon arrives, initially linearly polarised at angle θ. Say it is in a state of polarisation $|\theta\rangle$.

1. The photon either passes xor is absorbed.

2. The photon that passed is polarised along x-axis ($|x\rangle$). It has probability $\cos^2\theta$ of passing.

3. "Where" was that $|x\rangle$ photon before? We say that $|\theta\rangle$ was a linear superposition of $|x\rangle$ and $|y\rangle$ basis states, and the polariser selected $|x\rangle$ states.

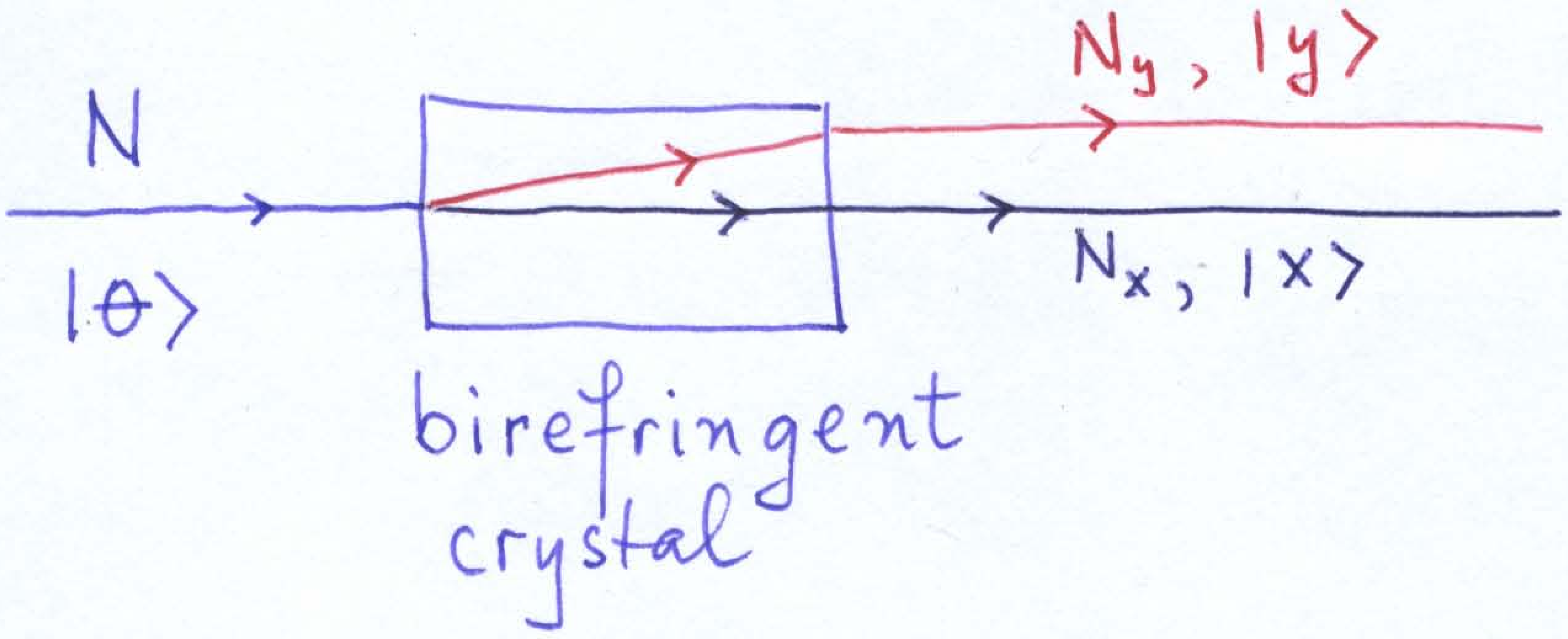
$$|\theta\rangle = \cos\theta\ |x\rangle + \sin\theta\ |y\rangle$$

$$P_{|x\rangle} = \cos^2\theta \quad ; \quad P_{|y\rangle} = \sin^2\theta$$

Among greatest achievements of $\overline{XX}$-century science

Photon as particle

$(h\nu = \hbar\omega, \ \hbar = \frac{h}{2\pi})$



$P_{|y\rangle} = \sin^2\theta \quad = N_y/N$

$P_{|x\rangle} = \cos^2\theta \quad = N_x/N$

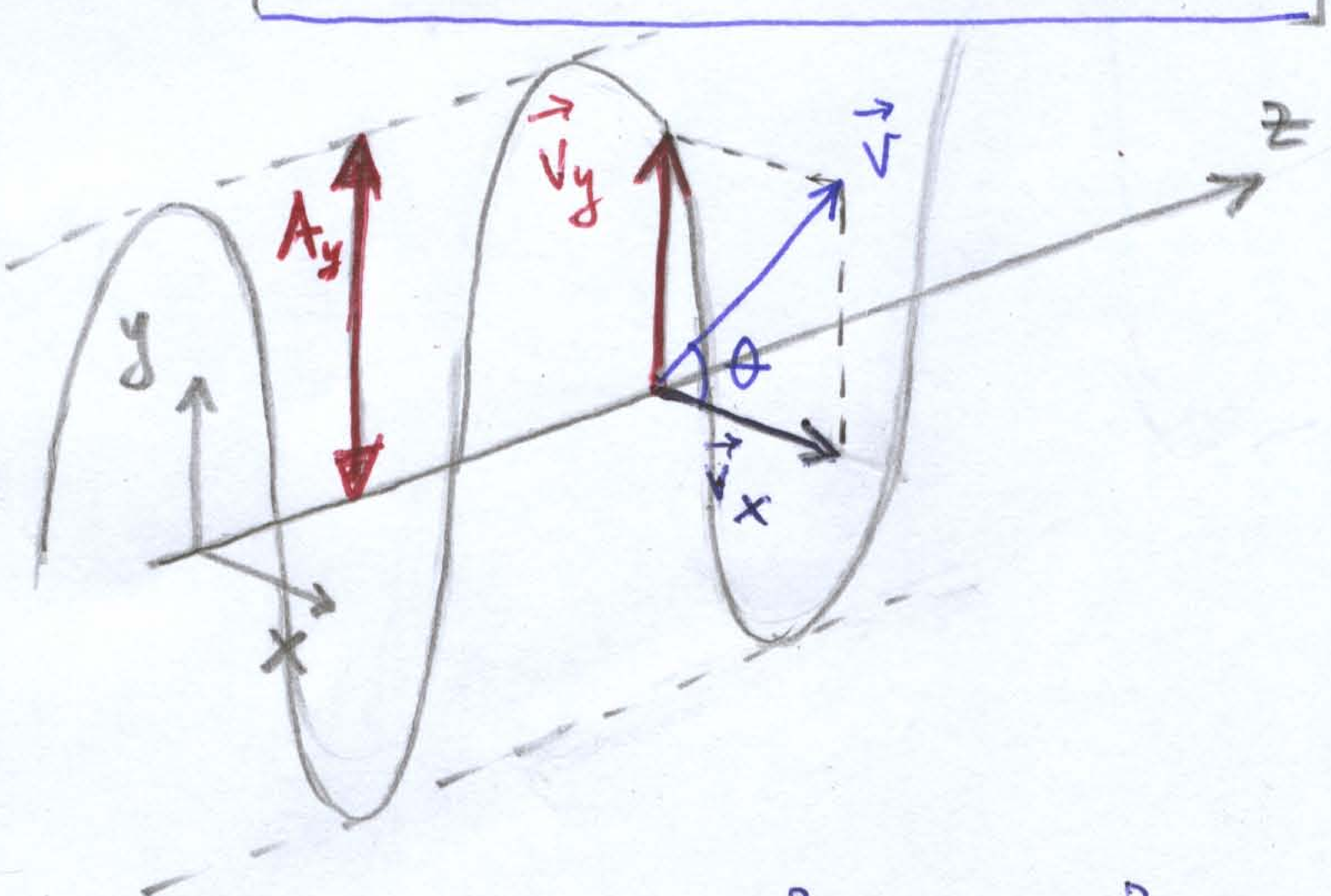
$1 = \frac{N_x}{N} + \frac{N_y}{N}$

$N = N_x + N_y$ (ok)

Energy $E = N\cdot(\hbar\omega)$

$\boxed{E \sim N}$

Photon as wave



$\vec{v} = \vec{v}_x + \vec{v}_y$

$= A_x \hat{x} \sin\omega t + A_y \hat{y} \sin\omega t$

$= (A_x \hat{x} + A_y \hat{y}) \sin\omega t$

$\Downarrow$

1 $\boxed{\frac{A_x}{A_y} = \frac{\cos\theta}{\sin\theta}}$

Energy $E \sim A^2 = A_x^2 + A_y^2$

$E = E_x + E_y$

$N = N_x + N_y$

$1 = \cos^2\theta + \sin^2\theta$

2 $\Rightarrow \boxed{A_x^2 \sim E_x \sim N_x \sim \cos^2\theta}$

Photon as a quantum state

(no more picture ...)

$$|\theta\rangle = \underset{\cos\theta}{A_x}\ \underbrace{|x\rangle}_{\omega,\text{ pol. no }\theta} + \underset{\sin\theta}{\boxed{A_y}}\ \underbrace{|y\rangle}_{\omega,\text{ pol. no }\theta}$$

Thus : $|\psi\rangle = c_1 |1\rangle + c_2 |2\rangle$, $c_1, c_2 \in \mathbb{C}$

if have only 2 basic states

(Once $|1\rangle$ chosen, take $|2\rangle$ such that it somehow excludes $|1\rangle$: being in one for sure excludes being in the other maximally.
("orthogonal states"))

$\langle 1|2\rangle = 0$ (like $\vec{v}\cdot\vec{w} = 0$, $\vec{v} \perp \vec{w}$)

If measure :

$$P_{|1\rangle} = |c_1|^2 , \quad P_{|2\rangle} = |c_2|^2 .$$

For $|\theta\rangle = \cos\theta\, |x\rangle + \sin\theta\, |y\rangle$,

observe that $P_{|x\rangle} + P_{|y\rangle} = 1$ (two states are enough)

Quantum mechanics :

1) What are the basis states ? - How many ?
- Which set to chose ?

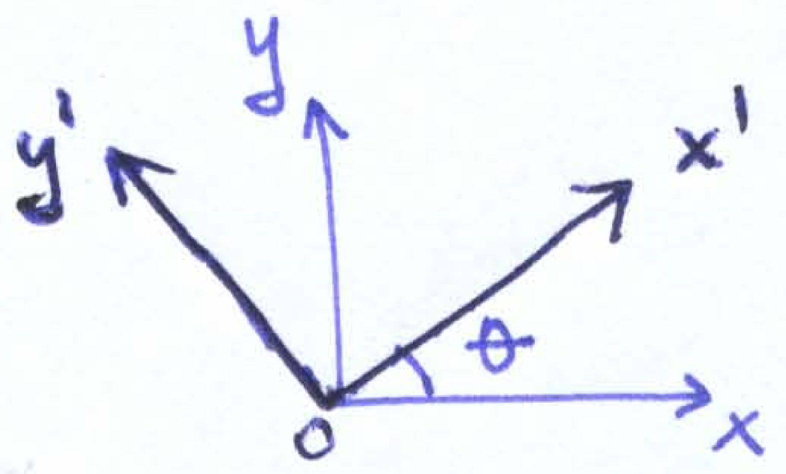
- massive particle with spin $\frac{1}{2}$: $\{|\uparrow\rangle, |\downarrow\rangle\}$
- particle on a linè $\{|x\rangle \,|\, x\in\mathbb{R}\}$
or $\{|p\rangle \,|\, p\in\mathbb{R}\}$.

2) Decompose $|\psi\rangle$ into basis states : $|\psi\rangle = \sum\!\!\!\!\!\!\int_i c_i |i\rangle$

3) $c_i(t) = ?$

Then, $P_i = |c_i(t)|^2$

Back to photons :



→ eg. polariser (detector) can have axis ∥ Ox'

Change of basis :
(Linear polarisation)
still

$$\begin{cases} |x'\rangle = \cos\theta \; |x\rangle + \sin\theta \; |y\rangle \\ |y'\rangle = -\sin\theta \; |x\rangle + \cos\theta \; |y\rangle \end{cases}$$

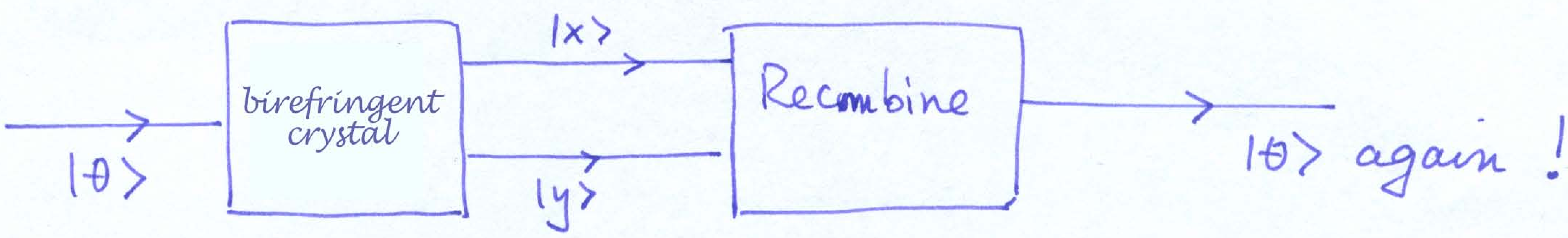
Right and left
Circular Polarisation
states :

$$\begin{cases} |R\rangle = \frac{1}{\sqrt{2}} |x\rangle + \frac{i}{\sqrt{2}} |y\rangle \\ |L\rangle = \frac{1}{\sqrt{2}} |x\rangle - \frac{i}{\sqrt{2}} |y\rangle \end{cases}$$

Note : if $|R'\rangle = \frac{|x'\rangle}{\sqrt{2}} + \frac{i}{\sqrt{2}} |y'\rangle$, $|L'\rangle = \frac{1}{\sqrt{2}} |x'\rangle - \frac{i}{\sqrt{2}} |y'\rangle$

then $|R'\rangle = \underbrace{e^{-i\theta}}_{\text{just a phase}} |R\rangle$, $|L'\rangle = \underbrace{e^{i\theta}}_{\text{the opposite phase}} |L\rangle$

Reconstruction of states :



- If measure in-between, no $|\theta\rangle$ anymore at the end!
- If do not measure, can't say anything about x or y ... If measure, get only intensity.

General again vs. VECTORS

$$\begin{cases} |\phi\rangle = \sum_i c_i |i\rangle \\ P_{|i\rangle} = |c_i|^2 \end{cases}$$

$$\vec{a} = \sum_i a_i \vec{e}_i = a_x \hat{x} + a_y \hat{y} + a_z \hat{z}$$

$$|\chi\rangle = \sum_j d_j |j\rangle$$

$$\vec{b} = \sum_i b_j \vec{e}_j$$

$$\langle j|i\rangle = 0 \quad \text{if } j \neq i$$

$$\langle i|i\rangle = 1$$

$$\vec{e}_i \cdot \vec{e}_j = 0 \quad \text{if } i \neq j$$

$$\vec{e}_i \cdot \vec{e}_i = 1$$

$$\langle \chi|\phi\rangle = \left(\sum_j d_j^* \langle j|\right)\left(\sum_i c_i |i\rangle\right) = \sum_i d_i^* c_i$$

$$\vec{a} \cdot \vec{b} = \sum_i a_i b_i$$

$$P(|\phi\rangle \to |\chi\rangle) = \sum_i d_i^* c_i = \langle \chi|\phi\rangle$$

$$P(|\phi\rangle \to |\phi\rangle) = \sum_i |c_i|^2 = 1$$

(OK)

Questions :

- What are the basis states in the problem ?
- What is the evolution of a given state, once we give ("prepare") it ?

Time evolution - general case

$$\langle x | \; U(t_2, t_1) \; | \phi \rangle = ? \qquad (\text{Amplitude} = ?)$$

measure WAIT prepare state

$$\langle x | U | \phi \rangle = \sum_i \sum_j \langle x | i \rangle \underbrace{\langle i | U | j \rangle}_{U_{ij}} \langle j | \phi \rangle$$

$$| \psi(t+\Delta t) \rangle = U(t+\Delta t, t) \, | \psi(t) \rangle$$

Q. $\begin{cases} \text{Given} \quad c_i(t) = \langle i | \psi(t) \rangle & \left(\text{i.e. } |\psi(t)\rangle = \sum_i c_i(t) \, |i\rangle \right) \\ \text{what is} \quad c_i(t+\Delta t)\,? & \left(\text{i.e. } |\psi(t+\Delta t)\rangle = \sum_i \underline{c_i(t+\Delta t)} \, |i\rangle \right) \end{cases}$

$$U_{ij} = \delta_{ij} + K_{ij} \, \Delta t = \delta_{ij} - \frac{i}{\hbar} H_{ij} \, \Delta t$$

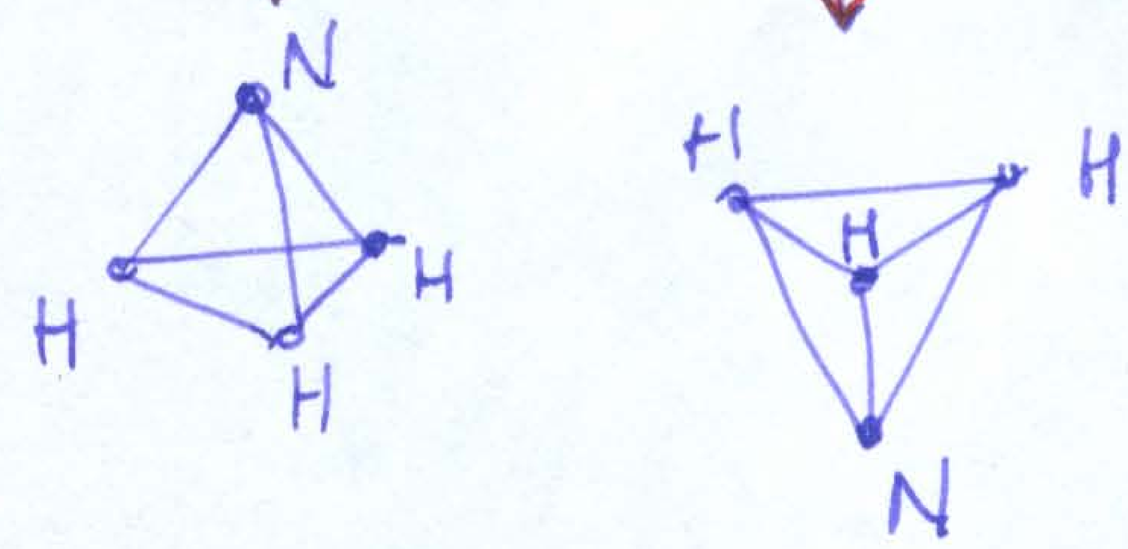
A. $\left\{ \; i\hbar \dfrac{dc_i}{dt} = \sum_j H_{ij}(t) \, c_j(t) \right.$

So:
1. Choose $\{|i\rangle\}$ ("basis")
2. Find $c_i(t)$, from $|\psi\rangle$: $c_i(t) = \langle i | \psi \rangle$ (at t_0 say)
3. Solve the set of differential equations, to find $c_i(t)$ for any t.

Time evolution - 2 states

$$|\psi\rangle = \langle 1|\psi\rangle \cdot |1\rangle + \langle 2|\psi\rangle \cdot |2\rangle = C_1|1\rangle + C_2|2\rangle.$$

$$\begin{cases} i\hbar \dfrac{dC_1}{dt} = H_{11} C_1 + H_{12} C_2 \\ i\hbar \dfrac{dC_2}{dt} = H_{21} C_1 + H_{22} C_2 \end{cases}$$



ammonia molecule
(no transl., no rot.)

$H_{ij} \neq 0$ for $i \neq j$: Transitions

1) $H_{12} = H_{21} = 0$: 2 independent equations :

$C_1(t) = C_1(0)\, e^{-i/\hbar\, H_{11}\, t}$

$|C_1(t)| = 1$ - ok - so H_{11} must be real (see why $-i/\hbar$ in front)

2) $H_{12} = H_{21} = -A$ (real, this time)

$H_{11} = H_{22} = E_0$ (some symmetry $|1\rangle \leftrightarrow |2\rangle$)

$$\Rightarrow \begin{cases} C_1(t) = \dfrac{a}{2} e^{-\frac{i}{\hbar}(E_0 - A)t} + \dfrac{b}{2} e^{-i/\hbar\,(E_0 + A)t} \\ C_2(t) = \dfrac{a}{2} e^{-i/\hbar\,(E_0 - A)t} - \dfrac{b}{2} e^{-i/\hbar\,(E_0 + A)t} \end{cases}$$

$a = 1, b = 0$: mode 1

$a = 0, b = 1$: mode 2

$a = 1 = b$: $C_1(0) = 1$, $\boxed{C_2(0) = 0}$ $\Rightarrow |C_2(t)|^2 = P_{|2\rangle} = \sin^2 \dfrac{At}{\hbar}$

2 - Pendulum analogy

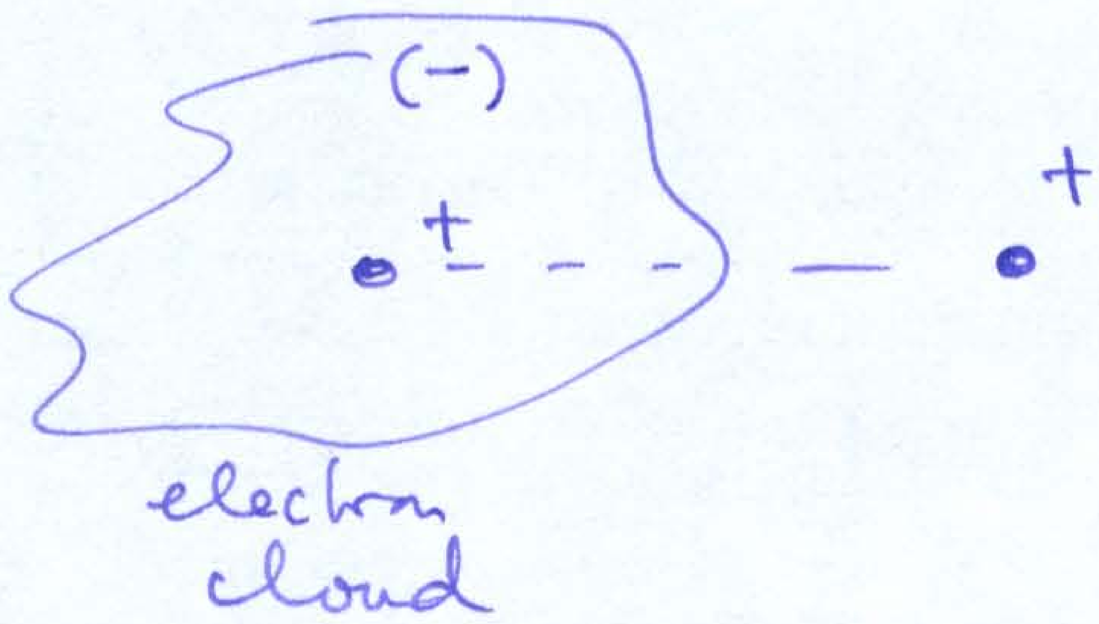
Particle in a magnetic field

$$H_{ij} = \begin{pmatrix} -\mu B_z & -\mu(B_x - iB_y) \\ -\mu(B_x + iB_y) & +\mu B_z \end{pmatrix}$$

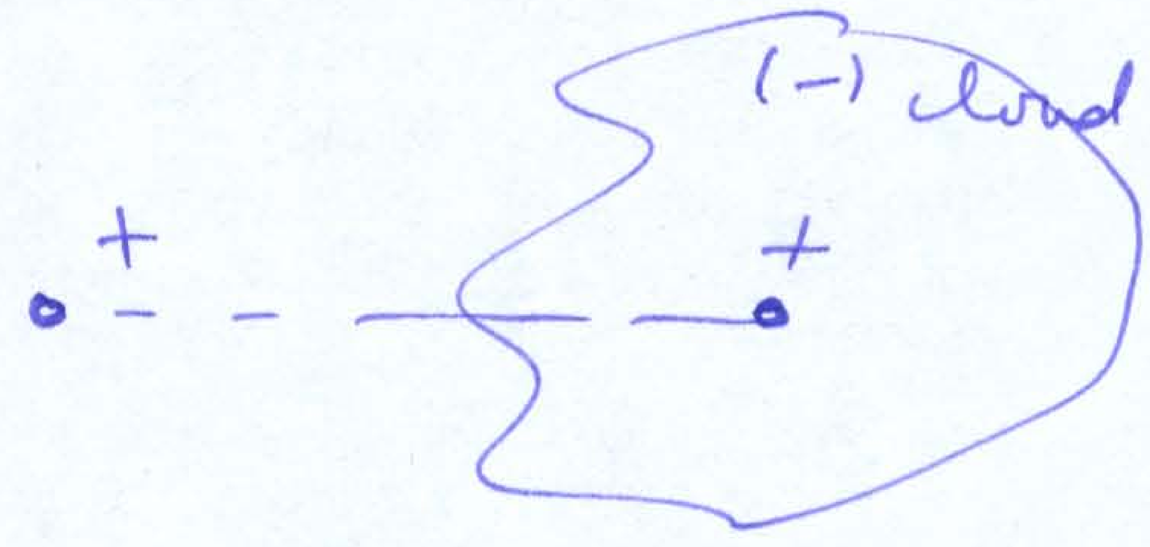
in $\{|+z\rangle, |-z\rangle\}$ basis.

- most general form linear in components of $\vec{B}$
- $H_{12} = H_{21}^* \neq H_{21}$ (not real)
- $H_{11} = -H_{22}$ due to choice of energy scale.

H_2^+



$\equiv |1\rangle$



$\equiv |2\rangle$

Again, neglect translations, rotations, vibrations.

Neutral K-mesons

$\bar{K}^0$, K^0 ; K^- , K^+ ($m \approx 500$ MeV $\sim \frac{m_{proton}}{2}$ $\sim 3\times m_\pi$)

($d\bar{s}$) ($d\bar{s}$) ($\bar{u}s$) ($u\bar{s}$)

strangeness: -1 , 1 , -1 , 1

↓

conserved by STRONG INTERACTIONS (10^{-23} s)

but not by WEAK INTERACTIONS (10^{-10} s).

$$K^0 \rightleftarrows \pi^- + \pi^+ \rightleftarrows \bar{K}^0$$

$$\boxed{\begin{aligned} C_+ &\equiv \langle K^0 | \psi \rangle \\ C_- &\equiv \langle \bar{K}^0 | \psi \rangle \end{aligned}}$$

$$\begin{cases} i\hbar \dfrac{dC_+}{dt} = E_0 C_+ + A C_- + A C_+ \\ i\hbar \dfrac{dC_-}{dt} = E_0 C_- + A C_+ + A C_- \end{cases}$$

A not real any more, due to decay possibility.

new states : $|K_1\rangle = \frac{1}{\sqrt{2}}|K_0\rangle + \frac{1}{\sqrt{2}}|\bar{K}_0\rangle$

$|K_2\rangle = \frac{1}{\sqrt{2}}|K_0\rangle - \frac{1}{\sqrt{2}}|\bar{K}_0\rangle$

$$\boxed{\begin{aligned} C_1 &\equiv \langle K_1 | \psi \rangle \\ C_2 &\equiv \langle K_2 | \psi \rangle \end{aligned}}$$

$$i\hbar \frac{dC_1}{dt} = 2AC_1 \Rightarrow C_1(t) = C_1(0)\, e^{-2iAt} = C_1(0)\, e^{-\beta t} e^{-i\alpha t}$$

$$i\hbar \frac{dC_2}{dt} = 0 \Rightarrow C_2(t) = C_2(0)$$

If start in $|K_1\rangle$, never get $|K_2\rangle$ (and $K_1 \to \pi^+\pi^-$)

If start in $|K_2\rangle$, stay so forever ($\tau \sim 600 \times 10^{-10}$ s, actually, due to $K_2 \to 3\pi$)